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Metode Matematika (C)

16. Buktikan $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$, $0<p<1$

* $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$

* $\Gamma(1-p) = \int_0^\infty x^{-p} e^{-x}\, dx$

Dari kedua bentuk tersebut, diperoleh :

$$\Gamma(p)\,\Gamma(1-p) = \int_0^\infty \int_0^\infty (x^{p-1}e^{-x})(y^{-p}e^{-y})\, dy\, dx$$

$$= \int_0^\infty \int_0^\infty \frac{1}{x}\cdot\left(\frac{x}{y}\right)^p e^{-(x+y)}\, dy\, dx$$

misal : $\dfrac{x}{y} = u$ $\quad x+y = v$

$$x = uy \longrightarrow uy + y = v$$
$$y(u+1) = v$$
$$y = \frac{v}{u+1}$$

$$y = v - x$$
$$x = uy \longrightarrow y = \frac{x}{u}$$
$$v - x = \frac{x}{u}$$
$$v = \frac{x}{u} + x$$
$$x\left(\frac{1}{u}+1\right) = v$$
$$x = \frac{v.u}{u+1}$$

$\Rightarrow$ $$\frac{\partial x}{\partial u} = \frac{v(u+1) - uv}{(u+1)^2} = \frac{uv - uv + v}{(u+1)^2} = \frac{v}{(u+1)^2}$$

$\Rightarrow$ $$\frac{\partial x}{\partial v} = \frac{u(u+1) - uv(0)}{(u+1)^2} = \frac{u^2+u}{(u+1)^2} = \frac{u\cancel{(u+1)}}{(u+1)^{\cancel{2}}} = \frac{u}{u+1}$$

$\Rightarrow$ $$\frac{\partial y}{\partial u} = \frac{0 - v}{(u+1)^2} = \frac{-v}{(u+1)^2}$$

$\Rightarrow$ $$\frac{\partial y}{\partial v} = \frac{u+1}{(u+1)^2} = \frac{1}{u+1}$$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{v}{(u+1)^2} & \dfrac{u}{u+1} \\ \dfrac{-v}{(u+1)^2} & \dfrac{1}{u+1} \end{vmatrix}$$

$$= \frac{v}{(u+1)^3} + \frac{uv}{(u+1)^3}$$

$$= \frac{v\cancel{(u+1)}}{\cancel{(u+1)^3}\,{}^{2}}$$

$$= \frac{v}{(u+1)^2}$$

Jadi, diperoleh :

$$\int_0^\infty \int_0^\infty u^p . e^{-v} . \frac{\cancel{v}}{\cancel{(u+1)^2}_{(u+1)}} . \frac{\cancel{u+1}}{u\cancel{v}} \, dv\, du$$

$$= \int_0^\infty \int_0^\infty \frac{u^{p-1}e^{-v}}{u+1}\, du\, dv$$

$$= \int_0^\infty \frac{u^{p-1}}{u+1}\, du \int_0^\infty e^{-v}\, dv$$

$$= * \int_0^\infty e^{-v}\, dv = \lim_{t\to\infty} \int_0^t e^{-v}\, dv$$
$$= \lim_{t\to\infty} -e^{-v}\Big]_0^t$$
$$= \lim_{t\to\infty} -e^{-t} + e^0$$
$$= \lim_{t\to\infty} -\frac{1}{e^t} + e^0$$
$$= 0 + 1$$
$$= 1$$

$$= \int_0^\infty \frac{u^{p-1}}{u+1}\, du = \frac{\pi}{\sin(\pi p)}$$ [TERBUKTI]

, $0<p<1$

17 $\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\,\Gamma(2x)$

$B(x,x) = \int_0^1 p^{x-1}(1-p)^{x-1}\,dp =$

$B(x,x) = \frac{\Gamma(x)\,\Gamma(x)}{\Gamma(2x)} = \frac{\Gamma^2(x)}{\Gamma(2x)}$

misal : $p = \sin^2 u$ BA : $\sin^2 u = 1$

$dp = 2\sin u\cos u\,du$ $u = \pi/2$

BB : $\sin^2 u = 0$

$u = 0$

$= \int_0^{\pi/2} (\sin u)^{2x-2}(1-\sin^2 u)^{x-1}\,2\sin u\cos u\,du$

$= 2\int_0^{\pi/2} (\sin u)^{2x-2}(\cos u)^{2x-2}\sin u\cos u\,du$

$= 2\int_0^{\pi/2} (\sin u)^{2x-2+1}(\cos u)^{2x-2+1}\,du$

$= 2\int_0^{\pi/2} (\sin u)^{2x-1}(\cos u)^{2x-1}\,du$

$= 2\int_0^{\pi/2} (\sin u\cos u)^{2x-1}\,du$

※ $\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{\sin 2\theta}{2} = \sin\theta\cos\theta$

$= 2\int_0^{\pi/2}\left(\frac{\sin 2u}{2}\right)^{2x-1}du$

$= \frac{2}{2^{2x-1}}\int_0^{\pi/2}(\sin 2u)^{2x-1}\,du$

misal : $2u = \theta \longrightarrow$ BA : $\theta = \pi$

$2du = d\theta \longrightarrow$ BB : $\theta = 0$

$= 2^{2-2x}\int_0^{\pi}(\sin\theta)^{2x-1}\,\frac{d\theta}{2}$

$= 2^{1-2x}\cdot\left[2\int_0^{\pi/2}(\sin(\theta))^{2x-1}(\cos\theta)^0\,d\theta\right]$

$=$ ※ $2m-1 = 2x-1$ ※ $2n-1 = 0$

$m = x$ $n = \frac{1}{2}$

$= 2^{1-2x}\,B\left(x,\frac{1}{2}\right)$

$= 2^{1-2x}\cdot\frac{\Gamma(x)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)}$

$= 2^{1-2x}\cdot\frac{\Gamma(x)\cdot\sqrt{\pi}}{\Gamma\left(x+\frac{1}{2}\right)}$

$\frac{\Gamma^2(x)}{\Gamma(2x)} = \frac{2^{1-2x}\cdot\cancel{\Gamma(x)}\sqrt{\pi}}{\Gamma\left(x+\frac{1}{2}\right)}$

$\Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,(\Gamma(2x))$

$\Rightarrow \Gamma(x)\,\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\,\Gamma(2x)$

// [TERBUKTI]

19). Buktikan $\int_0^{\pi/2}\left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{1/4}\cos x\,dx = \beta\left(\frac{1}{4},\frac{5}{4}\right)$

$= \int_0^{\pi/2}\left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{1/4}\cos x\,dx$

$= \int_0^{\pi/2}\left(\frac{1-\sin x}{\sin^3 x}\right)^{1/4}\cos x\,dx$

misal : $\sin x = p$ BA : $p = \sin\pi/2 = 1$

$\cos x\,dx = dp$ BB : $p = \sin 0 = 0$

$= \int_0^1\left(\frac{1-p}{p^3}\right)^{1/4}dp$

$= \int_0^1 (1-p)^{1/4}\,p^{-3/4}\,dp$

$= \int_0^1 p^{-3/4}(1-p)^{1/4}\,dp \longrightarrow$ Bentuk umum Fungsi Beta

※ $m-1 = -\frac{3}{4}$ ※ $n-1 = \frac{1}{4}$

$m = \frac{1}{4}$ $n = \frac{5}{4}$

$= \int_0^1 p^{-3/4}(1-p)^{1/4}\,dp = \beta\left(\frac{1}{4},\frac{5}{4}\right)$

[Terbukti]

$\int_0^{\pi/2}\left(\frac{1}{\sin^3 x} - \frac{1}{\sin^2 x}\right)^{1/4}\cos x\,dx = \beta\left(\frac{1}{4},\frac{5}{4}\right)$

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KIKY

21. Buktikan rumus Stirling berikut ini

$$n! = \sqrt{2\pi n}\; n^n e^{-n}$$

$$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

misal : $x = nt$ BA : $t = \infty$

$dx = n dt$ BB : $t = 0$

$$= \int_0^\infty (nt)^n e^{-nt} dt$$

$$= n^{n+1} \int_0^\infty t^n . e^{-nt} dt$$

$$= n^{n+1} \int_0^\infty e^{\ln t^n} e^{-nt} dt$$

$$= n^{n+1} \int_0^\infty e^{n\ln t - nt} dt$$

$$= n^{n+1} \int_0^\infty e^{n(\ln t - 1)} dt$$

※ Aproksimasi Deret Taylor :

$$\boxed{\ln(t) - t \approx -1 - \frac{(t-1)^2}{2}}$$

$$= n^{n+1} \int_0^\infty e^{n\left(-1 - \frac{(t-1)^2}{2}\right)} dt$$

$$= n^{n+1} . e^{-n} \int_0^\infty e^{-\frac{n(t-1)^2}{2}} dt$$

$$= n^{n+1} e^{-n} \int_0^\infty e^{-\sqrt{\frac{n}{2}}(t-1)} dt$$

misal: $p = \frac{\sqrt{n}\,(t-1)}{\sqrt{2}} \rightarrow dp = \frac{\sqrt{n}}{\sqrt{2}} dt$

$dt = \frac{\sqrt{2}}{\sqrt{n}} dp$

BA $= \infty$ BB $= -\frac{\sqrt{n}}{\sqrt{2}}$

$$\Rightarrow n^{n+1} e^{-n} \int_{-\sqrt{\frac{n}{2}}}^{\infty} e^{-p^2} \sqrt{\frac{2}{n}}\, dp$$

$$= \frac{n^{n+1} e^{-n}}{\sqrt{n}} . \sqrt{2} . n \int_{-\sqrt{\pi/2}}^{\infty} e^{-p^2} dp$$

$$= n^n e^{-n} \sqrt{2n}\; n \int_{-\sqrt{\pi/2}}^{\infty} e^{-p^2} dp$$

Untuk n sangat besar $(n \to \infty)$ $-\frac{\sqrt{\pi}}{\sqrt{2}} \Rightarrow -\infty$

$$= n^n e^{-n} \sqrt{2n} \int_{-\infty}^{\infty} e^{-p^2} dp$$

$$= n^n e^{-n} \sqrt{2n}\; (\sqrt{\pi})$$

$$= n^n e^{-n} \sqrt{2n\pi}$$ [TERBUKTI]

24) Tunjukkan $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{x}+1}\, dx = \frac{\pi}{\sin \pi}$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{e^{x}+1}\, dx$$

misal : $e^{x} = p$ $\qquad$ BA : $p = \infty$

$e^{x}\, dx = dp$ $\qquad$ BB : $p = 0$

$dx = \frac{dp}{p}$

$$\int_{0}^{\infty} \frac{p^{a}}{p+1} \frac{dp}{p} = \int_{0}^{\infty} \frac{p^{a-1}}{p+1}\, dp$$

$$= \Gamma(a)\, \Gamma(1-a)$$

$$= \frac{\pi}{\sin \pi a} \quad /\!/ \quad [\text{TERBUKTI}]$$